

Orbital motions as gradiometers for post-Newtonian tidal effects

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ABSTRACT

The direct long-term changes occurring in the orbital dynamics of a local gravitationally bound binary system S due to the post-Newtonian tidal acceleration caused by an external massive source are investigated. A class of systems made of a test particle m rapidly orbiting with orbital frequency n_b an astronomical body of mass M which, in turn, slowly revolves around a distant object of mass M' with orbital frequency $n'_b \ll n_b$ is considered. The characteristic frequencies of the non-Keplerian orbital variations of m and of M itself are assumed to be negligible with respect to both n_b and n'_b . General expressions for the resulting Newtonian and post-Newtonian tidal orbital shifts of m are obtained. The future missions BepiColombo and JUICE to Mercury and Ganymede, respectively, are considered in view of a possible detection. The largest effects, of the order of $\approx 0.1 - 0.5$ milliarcseconds per year (mas yr^{-1}), occur for the Ganymede orbiter of the JUICE mission. Although future improvements in spacecraft tracking and orbit determination might, perhaps, reach the required sensitivity, the systematic bias represented by the other known orbital perturbations of both Newtonian and post-Newtonian origin would be overwhelming. The realization of a dedicated artificial mini-planetary system to be carried onboard and Earth-orbiting spacecraft is considered as well. Post-Newtonian tidal precessions as large as $\approx 1 - 10^2 \text{ mas yr}^{-1}$ could be obtained, but the quite larger Newtonian tidal effects would be a major source of systematic bias because of the present-day percent uncertainty in the product of the Earth's mass times the Newtonian gravitational parameter.

Keywords: General relativity and gravitation; Experimental studies of gravity; Experimental tests of gravitational theories; Celestial mechanics; Spacecraft

1 INTRODUCTION

Gravitation is one of the known fundamental interactions of physics, and the General Theory of Relativity (GTR) is, at present, its best theoretical description [1]. As such, GTR is one of the pillars of our knowledge of Nature; intense experimental and observational scrutiny is required not only to gain an ever-increasing confidence about it, but also to explore the borders of the realm of its validity at different scales. To this aim, a variety of different theoretical, experimental and observational approaches are required to extend the frontiers of our knowledge of the gravitational phenomena. Are there some founded hopes to testing newly predicted gravitational effects in the near future in some suitable astronomical and astrophysical laboratories? What are the possibilities opened up by forthcoming space-based missions?

The present paper will try to address these questions by looking at certain effects that the components of the Riemann spacetime curvature tensor are expected to induce on local systems according to GTR.

The internal dynamics of a gravitationally bound binary system immersed in the external gravitational field of a massive rotating body is tidally affected at both the Newtonian and the post-Newtonian level [2, 3, 4, 5]. In this paper, we will look in detail at some of the post-Newtonian orbital effects of tidal origin arising in the relative motion of a restricted two-body system, and at the possibility of detecting them in either natural or artificial space-based scenarios. Special cases widely treated in the literature are the post-Newtonian tidal effects of the rotating Sun's field in the Earth-Moon system [6, 7, 8, 9, 10, 11] and of the spinning Earth itself in arrays of spaceborne artificial gradiometers [7, 12, 13, 14, 15, 16, 17]. Our calculation will have a broad range of validity. Indeed, while, on the one hand, certain assumptions on the characteristic orbital frequencies of the three-body system considered will be necessarily made, on the other hand, we will remove the limitations existing in the literature [4] on either the orientation of the spin axes of the external objects and on the orbital configurations of the moving bodies.

The paper is organized as follows. In Section 2, the long-term rates of change of the orbital parameters of the test particle of the restricted two-body system are calculated by keeping the elements of a generic tidal matrix constant over the orbital period of the particle around its primary. In Section 3, the direct orbital effects due to both the gravitoelectric and the gravitomagnetic tidal matrices are obtained by averaging their elements over the orbital period of the motion around the distant body. Section 4 is devoted to exploring some experimental possibilities offered by forthcoming spacecraft-based missions to astronomical bodies. Section 5 summarizes our findings.

2 THE LONG-TERM ORBITAL RATES OF CHANGE AVERAGED OVER P_B

Let us consider an isolated rotating body of mass M' , equatorial radius R' and proper angular momentum \mathbf{J}' at rest in some parameterized post-Newtonian coordinate system \mathcal{K}' whose spatial axes point to distant stars; as such, \mathcal{K}' is kinematically and dynamically non-rotating [18]. Let a local gravitationally bound system S move geodesically around M' ; S is assumed to be made of a body of mass M , equatorial radius R and proper angular momentum \mathbf{J} , and of a test particle of mass m revolving about M itself. For the sake of simplicity, we will assume $m \ll M \ll M'$; examples of such scenarios are spacecraft like MESSENGER [19] and the forthcoming BepiColombo [20] to orbit Mercury in the field of the Sun as well as the future JUICE mission [21] to orbit Ganymede and to study the Jovian system. To the Newtonian level, the spatial trajectory of S about M' can be parameterized in terms of the usual Keplerian orbital elements. They are the semimajor axis a' , the eccentricity e' , the inclination I' to the reference $\{x', y'\}$ plane of \mathcal{K}' , the longitude of the ascending node Ω' , and the argument of pericenter ω' ; $n'_b = \sqrt{GM'a'^{-3}}$ is the Keplerian orbital frequency, where G is the Newtonian constant of gravitation. In general, such orbital parameters do not stay constant because of the well known Newtonian and post-Newtonian departures from spherical symmetry of the field of M' . They induce the post-Newtonian Einstein (gravitoelectric) [22, 23] and Lense-Thirring (gravitomagnetic) [24] orbital precessions as well as the classical ones due to the oblateness J'_2 of M' and, possibly, to its other multipoles of higher order [25].

Let a local inertial frame \mathcal{K} , attached to M , be parallel transported along the geodesic worldline of M through the spacetime of M' [26, 27, 28]. As such, the spatial axes of \mathcal{K} change naturally their orientation with respect to the fixed ‘‘Copernican’’ spatial axes of \mathcal{K}' because of the geodesic motion of \mathcal{K} itself through the external spacetime deformed by M' and \mathbf{J}' . As such, \mathcal{K} is also said to be kinematically rotating because of the resulting de Sitter-Fokker (gravitoelectric) [29, 30] and Pugh-Schiff (gravitomagnetic)

precessions¹ of its axes with respect to those of \mathcal{K}' , but it is dynamically nonrotating because of the absence of Coriolis and centrifugal inertial forces [18]. To the Newtonian level, also the motion of m about M will be parameterized in terms of a set of Keplerian orbital elements a, e, I, Ω, ω in such a way that $n_b = \sqrt{GMa^{-3}}$ denotes its Keplerian orbital frequency. Usually, the frequencies of the de Sitter-Fokker and Pugh-Schiff precessions are quite smaller than both n'_b and n_b ; for a critical discussion tidal phenomena occurring in the Sun-Earth-Moon system over timescales comparable to or larger than the de Sitter-Fokker and Pugh-Schiff ones, see [9, 10]. Thus, we can safely assume \mathcal{K} as kinematically nonrotating over timescales comparable to the orbital periods $P_b = 2\pi n_b^{-1}$ and $P'_b = 2\pi n'^{-1}_b$ of the three-body system considered. As a further assumption, we will consider the local motion of m about M much faster than the one of S itself around M' , i.e. $n'_b \ll n_b$. In general, the internal dynamics of S is not purely Keplerian because of possible departures from sphericity of M and of the post-Newtonian components of the field of M . As such, the orbit of m with respect to M undergoes the well-known Newtonian and post-Newtonian orbital precessions. As it occurs in the systems considered here, the timescales of such changes are quite longer than the orbital periods P_b, P'_b , i.e. $\dot{\Psi} \ll n'_b$, where Ψ denotes a generic precessing osculating Keplerian orbital element of m .

At both the Newtonian and the post-Newtonian level, the internal dynamics of S is locally affected also by tidal effects due to its motion through the external deformed spacetime of M' .

The tidal acceleration experienced by m is of the form [13]

$$\mathbf{A}_{\text{tid}} = -\mathbf{K}\mathbf{r}, \quad (1)$$

where the elements of the tidal matrix \mathbf{K}

$$K_{ij} = \mathcal{R}_{0i0j}, \quad i, j = 1, 2, 3 \quad (2)$$

are the tetrad components of the curvature Riemann tensor evaluated onto the geodesic of the observer in \mathcal{K} , and have dimensions of T^{-2} . It is

$$\mathbf{K} = \mathbf{K}^{(\text{N})} + \mathbf{K}^{(\text{GE})} + \mathbf{K}^{(\text{GM})}, \quad (3)$$

with the Newtonian (N), gravitoelectric (GE) and gravitomagnetic (GM) tidal matrices given by [13]

$$K_{ij}^{(\text{N})} = \frac{GM'}{r'^3} \left(\delta_{ij} - 3\hat{r}'_i \hat{r}'_j \right), \quad (4)$$

$$\begin{aligned} K_{ij}^{(\text{GE})} = & -\frac{G^2 M'^2}{c^2 r'^4} \left(3\delta_{ij} - 9\hat{r}'_i \hat{r}'_j \right) + \\ & + \frac{GM'}{c^2 r'^3} \left\{ 3 \left[v'^2 \delta_{ij} - v'_i v'_j + 3 \left(\mathbf{v}' \cdot \hat{\mathbf{r}}' \right) \hat{r}'_{(i} v'_{j)} \right] - 3 \left(\mathbf{v}' \cdot \hat{\mathbf{r}}' \right)^2 \delta_{ij} - 6\hat{r}'_i \hat{r}'_j v'^2 \right\}, \quad (5) \end{aligned}$$

$$K_{ij}^{(\text{GM})} = -\frac{6GJ'}{c^2 r'^4} \left\{ 3 \left(\mathbf{v}' \times \hat{\mathbf{k}}' \right)_{(i} \hat{r}'_{j)} + \left(\hat{\mathbf{r}}' \times \hat{\mathbf{k}}' \right)_{(i} v'_{j)} + \hat{\mathbf{r}}' \cdot \left(\mathbf{v}' \times \hat{\mathbf{k}}' \right) \left(\delta_{ij} - 5\hat{r}'_i \hat{r}'_j \right) - \right.$$

¹ A comoving coordinate system is said to be kinematically nonrotating if it is corrected for the post-Newtonian precessions of its axes.

$$- 5 \left(\hat{\mathbf{r}}' \cdot \mathbf{v}' \right) \left(\hat{\mathbf{r}}' \times \hat{\mathbf{k}}' \right)_{(i \hat{r}' j)} \Bigg\}. \quad (6)$$

In eq. (4)-eq. (6), which are symmetric and traceless, c is the speed of light in vacuum, $\hat{\mathbf{r}}' = \mathbf{r}'/r'$ is the versor of the position vector \mathbf{r}' from M' to M , \mathbf{v}' is the velocity vector of M with respect to M' , δ_{ij} is the Kronecker symbol, $\hat{\mathbf{k}}'$ is the unit vector of the spin axis of M' , the symbols \cdot and \times denote the usual scalar and cross products among vectors, and parentheses around indices denote symmetrization.

The tidal acceleration of eq. (1) can be considered as a small perturbation \mathbf{A}_{pert} of the Newtonian monopole of M . As such, its impact on the orbital dynamics of m can be treated perturbatively with standard techniques. By recalling the condition $n'_b \ll n_b$, the elements of the tidal matrix \mathbf{K} can be considered as constant over an orbital period P_b . Thus, by evaluating the right-hand-sides of the Gauss equations [31]

$$\frac{da}{dt} = \frac{2}{n_b \sqrt{1-e^2}} \left(e A_R \sin f + \frac{p}{r} A_T \right), \quad (7)$$

$$\frac{de}{dt} = \frac{\sqrt{1-e^2}}{n_b a} \left\{ A_R \sin f + A_T \left[\cos f + \frac{1}{e} \left(1 - \frac{r}{a} \right) \right] \right\}, \quad (8)$$

$$\frac{dI}{dt} = \frac{\cos(\omega + f)}{n_b a \sqrt{1-e^2}} \left(\frac{r}{a} \right) A_N, \quad (9)$$

$$\frac{d\Omega}{dt} = \frac{\sin(\omega + f)}{n_b a \sin I \sqrt{1-e^2}} \left(\frac{r}{a} \right) A_N, \quad (10)$$

$$\frac{d\omega}{dt} + \cos I \frac{d\Omega}{dt} = \frac{\sqrt{1-e^2}}{n_b a e} \left[-A_R \cos f + A_T \left(1 + \frac{r}{p} \right) \sin f \right]. \quad (11)$$

onto the unperturbed Keplerian ellipse

$$r = \frac{p}{1 + e \cos f}, \quad (12)$$

where $p = a(1 - e^2)$ is the semilatus rectum and A_R, A_T, A_N are the radial, transverse and out-of-plane components of eq. (1), the orbital variations of m averaged over P_b can be calculated. To this aim, let us note that, in principle, the average should be made by means of [32, 33]

$$\frac{df}{dt} = n_b \left(\frac{a}{r} \right)^2 \sqrt{1-e^2} - \left(\frac{d\omega}{dt} + \cos I \frac{d\Omega}{dt} \right), \quad (13)$$

where t is the proper time along the observer's geodesic, f is the true anomaly, and $\dot{\Omega}, \dot{\omega}$ are to be intended as the instantaneous, non-averaged precessions of the node and the pericenter [32]. Indeed, f is reckoned from the pericenter position, which, in general, changes because of possible variations of Ω and ω due to the non-Keplerian Newtonian and post-Newtonian effects within S . As such, the instantaneous

expressions for $\dot{\Omega}$ and $\dot{\omega}$ in eq. (13) should be taken from eq. (10)-eq. (11) themselves evaluated for the specific Newtonian and post-Newtonian perturbations [32]. However, by limiting ourselves just to the order $\mathcal{O}(c^{-2})$, and by neglecting small mixed terms of order $\mathcal{O}(J_2 c^{-2})$ arising from the interplay between the external tidal and the local Newtonian perturbations due to J_2 , the approximate expression

$$\frac{df}{dt} = n_b \left(\frac{a}{r}\right)^2 \sqrt{1 - e^2} \quad (14)$$

can be used by integrating over f between 0 and 2π . By using the following Keplerian expressions for the position \mathbf{r}

$$x = r [\cos \Omega \cos (f + \omega) - \cos I \sin \Omega \sin (f + \omega)], \quad (15)$$

$$y = r [\sin \Omega \cos (f + \omega) + \cos I \cos \Omega \sin (f + \omega)], \quad (16)$$

$$z = r [\sin I \sin (f + \omega)], \quad (17)$$

and the velocity \mathbf{v}

$$v_x = -\frac{an_b \{\cos I \sin \Omega [e \cos \omega + \cos (f + \omega)] + \cos \Omega [e \sin \omega + \sin (f + \omega)]\}}{\sqrt{1 - e^2}}, \quad (18)$$

$$v_y = \frac{an_b \{\cos I \cos \Omega [e \cos \omega + \cos (f + \omega)] - \sin \Omega [e \sin \omega + \sin (f + \omega)]\}}{\sqrt{1 - e^2}}, \quad (19)$$

$$v_z = \frac{an_b \sin I [e \cos \omega + \cos (f + \omega)]}{\sqrt{1 - e^2}}, \quad (20)$$

it is possible to compute from eq. (15)-eq. (20) the unit vector $\hat{\mathbf{L}}$ along the orbital angular momentum as

$$\hat{\mathbf{L}} = \frac{\mathbf{r} \times \mathbf{v}}{|\mathbf{r} \times \mathbf{v}|}. \quad (21)$$

Then, the radial, transverse and normal components of eq. (1) turn out to be

$$\begin{aligned} A_R = \mathbf{A}_{\text{tid}} \cdot \hat{\mathbf{r}} = & \frac{Ka(1 - e^2)}{1 + e \cos f} \{ [\cos I \sin (f + \omega) \sin \Omega - \cos (f + \omega) \cos \Omega] \cdot \\ & \cdot [\sin f (K_{21} \cos I \cos \omega \cos \Omega + (K_{22} + K_{33}) \sin \omega \cos \Omega + K_{31} \cos \omega \sin I) + \\ & + \cos f (K_{11} \cos \omega \cos \Omega + (K_{21} \cos I \cos \Omega + K_{31} \sin I) \sin \omega) + \\ & + (K_{21} \cos (f + \omega) + (K_{22} + K_{33}) \cos I \sin (f + \omega)) \sin \Omega] + \\ & + (\cos I \cos \Omega \sin (f + \omega) + \cos (f + \omega) \sin \Omega) (\cos (f + \omega) (K_{21} \cos \Omega + K_{22} \sin \Omega) + \end{aligned}$$

$$\begin{aligned}
& + \sin(f + \omega) (K_{32} \sin I + \cos I (K_{22} \cos \Omega - K_{21} \sin \Omega))) + \\
& + \sin I \sin(f + \omega) (\cos(f + \omega) (K_{31} \cos \Omega + K_{32} \sin \Omega) + \\
& + \sin(f + \omega) (K_{33} \sin I + \cos I (K_{32} \cos \Omega - K_{31} \sin \Omega))) \}, \tag{22}
\end{aligned}$$

$$\begin{aligned}
A_T = \mathbf{A}_{\text{tid}} \cdot (\hat{\mathbf{L}} \times \hat{\mathbf{r}}) = & -\frac{Ka(1-e^2)}{4(1+e \cos f)} \{(-K_{33} + (2K_{22} + K_{33}) \cos 2\Omega - \\
& - 2K_{21} \sin 2\Omega) \cos^2 I \sin(2f + 2\omega) + 2(2 \sin I (K_{32} \cos \Omega - K_{31} \sin \Omega) \sin(2f + 2\omega) + \\
& + [2K_{21} \cos 2\Omega + (2K_{22} + K_{33}) \sin 2\Omega]) \cos I \cos(2f + 2\omega) + \\
& + 4 \sin I (K_{31} \cos \Omega + K_{32} \sin \Omega) \cos(2f + 2\omega) + [(2 - \cos 2I) K_{33} + \\
& + (2K_{22} + K_{33}) \cos 2\Omega - 2K_{21} \sin 2\Omega] \sin(2f + 2\omega)\}, \tag{23}
\end{aligned}$$

$$\begin{aligned}
A_N = \mathbf{A}_{\text{tid}} \cdot \hat{\mathbf{L}} = & -\frac{Ka(1-e^2)}{4(1+e \cos f)} \{4 \cos I (K_{31} \cos \Omega + K_{32} \sin \Omega) \cos(f + \omega) + \\
& + 2 \sin I [-2K_{21} \cos 2\Omega - (2K_{22} + K_{33}) \sin 2\Omega] \cos(f + \omega) + \\
& + 4 \cos 2I (K_{32} \cos \Omega - K_{31} \sin \Omega) \sin(f + \omega) + \\
& + \sin 2I [3K_{33} - (2K_{22} + K_{33}) \cos 2\Omega + 2K_{21} \sin 2\Omega] \sin(f + \omega)\}, \tag{24}
\end{aligned}$$

where K is the dimensional scaling factor of the tidal matrix considered having dimensions of T^{-2} , while the dimensionless coefficients K_{ij} , $i, j = 1, 2, 3$ depend only on the orbital parameters of M and on the orientation of $\hat{\mathbf{k}}'$. Thus, by inserting eq. (22)-eq. (24) in the right-hand-sides of eq. (7)-eq. (11) and averaging them over P_b with eq. (14), one finally has

$$\left\langle \frac{da}{dt} \right\rangle_{P_b} = 0, \tag{25}$$

$$\left\langle \frac{de}{dt} \right\rangle_{P_b} = \frac{5Ke\sqrt{1-e^2}}{8n_b} \{-2 \sin 2\omega \sin \Omega (2K_{21} \cos \Omega + K_{33} \sin \Omega) \cos^2 I +$$

$$\begin{aligned}
& + 2 [2 \sin I \sin 2\omega (K_{32} \cos \Omega - K_{31} \sin \Omega) + \cos 2\omega (2K_{21} \cos 2\Omega + \\
& + (2K_{22} + K_{33}) \sin 2\Omega)] \cos I + 4 \cos 2\omega \sin I (K_{31} \cos \Omega + K_{32} \sin \Omega) + \\
& + \sin 2\omega [K_{22} (\cos 2I + 3) \cos 2\Omega + K_{33} (-\cos 2I + \cos 2\Omega + 2) - \\
& - 2K_{21} \sin 2\Omega] \}, \tag{26}
\end{aligned}$$

$$\begin{aligned}
\left\langle \frac{dI}{dt} \right\rangle_{P_b} &= -\frac{K}{8n_b \sqrt{1-e^2}} \{ 10e^2 \cos 2I \sin 2\omega (K_{32} \cos \Omega - K_{31} \sin \Omega) - \\
& - \frac{5}{2} e^2 \sin 2I \sin 2\omega [-3K_{33} + (2K_{22} + K_{33}) \cos 2\Omega - 2K_{21} \sin 2\Omega] + \\
& + 2 \cos I (5e^2 \cos 2\omega + 3e^2 + 2) (K_{31} \cos \Omega + K_{32} \sin \Omega) - \\
& - (5e^2 \cos 2\omega + 3e^2 + 2) \sin I [2K_{21} \cos 2\Omega + (2K_{22} + K_{33}) \sin 2\Omega] \}, \tag{27}
\end{aligned}$$

$$\begin{aligned}
\left\langle \frac{d\Omega}{dt} \right\rangle_{P_b} &= -\frac{K \csc I}{16n_b \sqrt{1-e^2}} \{ 20e^2 \sin 2\omega (\cos I (K_{31} \cos \Omega + K_{32} \sin \Omega) - \\
& - \sin I (K_{21} \cos 2\Omega + (2K_{22} + K_{33}) \cos \Omega \sin \Omega)) + \\
& + 4 \cos 2I (5e^2 \cos 2\omega - 3e^2 - 2) (K_{31} \sin \Omega - K_{32} \cos \Omega) + \\
& + (5e^2 \cos 2\omega - 3e^2 - 2) \sin 2I [-3K_{33} + (2K_{22} + K_{33}) \cos 2\Omega - \\
& - 2K_{21} \sin 2\Omega] \}, \tag{28}
\end{aligned}$$

$$\begin{aligned}
\left\langle \frac{d\omega}{dt} \right\rangle_{P_b} + \cos I \left\langle \frac{d\Omega}{dt} \right\rangle_{P_b} &= \frac{K \sqrt{1-e^2}}{16n_b} \{ -40 \sin I \sin 2\omega (K_{31} \cos \Omega + K_{32} \sin \Omega) + \\
& + 4 (5 \cos 2\omega - 3) \sin 2I (K_{32} \cos \Omega - K_{31} \sin \Omega) -
\end{aligned}$$

$$\begin{aligned}
& - 20 \cos I \sin 2\omega [2K_{21} \cos 2\Omega + (2K_{22} + K_{33}) \sin 2\Omega] + \\
& + \cos 2I (5 \cos 2\omega - 3) [-3K_{33} + (2K_{22} + K_{33}) \cos 2\Omega - 2K_{21} \sin 2\Omega] + \\
& + 3 (5 \cos 2\omega + 1) [K_{33} + (2K_{22} + K_{33}) \cos 2\Omega - 2K_{21} \sin 2\Omega] \}. \quad (29)
\end{aligned}$$

The long-term rates of eq. (25)-eq. (29) are valid for any symmetric and traceless tidal-type perturbation of the form of eq. (1) whose coefficients can be considered as constant over the characteristic orbital frequency n_b of the local binary system considered. As such, eq. (25)-eq. (29) are not limited just to eq. (5)-eq. (6). Moreover, eq. (25)-eq. (29) hold for a general orbital configuration of the test particle m since no a priori simplifying assumptions concerning its eccentricity and inclination were made. As shown by eq. (26), the eccentricity of a circular orbit is not affected by a tidal-type perturbation.

3 THE LONG-TERM ORBITAL RATES OF CHANGE AVERAGED OVER P'_B

In general, eq. (25)-eq. (29) may not be regarded as truly secular rates over timescales arbitrarily long because of the slow time dependence encoded in both the tidal matrix elements themselves and in the orbital elements of m , collectively denoted as $\{\Psi\}$, due to possible non-Keplerian local effects taking place in the non-spherically symmetric field of M .

Let us, now, assume that the characteristic timescales P_Ψ of all the non-Keplerian orbital effects within S are much longer than the orbital period P'_b of S itself about M' . It is a reasonable tenet, satisfied in several astronomical scenarios of potential experimental interest. Thus, it is possible to perform a further average of eq. (25)-eq. (29) over P'_b by keeping a, e, I, Ω, ω constant over the integration with respect to some fast variable² of the motion of M around M' .

The direct effects of order $\mathcal{O}(c^{-2})$ can be obtained by evaluating the post-Newtonian tidal matrices eq. (5)-eq. (6) onto an unchanging Keplerian ellipse as reference unperturbed trajectory.

Below, the averaged tidal matrix elements of eq. (4)-eq. (6), computed to order $\mathcal{O}(c^{-2})$ and to zero order in J'_2 , are listed. They are to be inserted in eq. (25)-eq. (29) to have the direct long-term rates of change of m averaged over P'_b .

As far as the Newtonian tidal matrix of eq. (4) is concerned, its average, to the zero order in J'_2 , is

$$\left\langle K_{11}^{(N)} \right\rangle_{P'_b} = - \frac{GM' \left(1 + 3 \cos 2I' + 6 \sin^2 I' \cos 2\Omega' \right)}{8a'^3 \left(1 - e'^2 \right)^{3/2}}, \quad (30)$$

$$\left\langle K_{22}^{(N)} \right\rangle_{P'_b} = - \frac{GM' \left(1 + 3 \cos 2I' - 6 \sin^2 I' \cos 2\Omega' \right)}{8a'^3 \left(1 - e'^2 \right)^{3/2}}, \quad (31)$$

² It turned out computationally more convenient to adopt the true anomaly f' .

$$\left\langle K_{33}^{(N)} \right\rangle_{P'_b} = \frac{GM' (1 + 3 \cos 2I')}{4a'^3 (1 - e'^2)^{3/2}}, \quad (32)$$

$$\left\langle K_{12}^{(N)} \right\rangle_{P'_b} = -\frac{3GM' \sin^2 I' \sin 2\Omega'}{4a'^3 (1 - e'^2)^{3/2}}, \quad (33)$$

$$\left\langle K_{13}^{(N)} \right\rangle_{P'_b} = \frac{3GM' \sin 2I' \sin \Omega'}{4a'^3 (1 - e'^2)^{3/2}}, \quad (34)$$

$$\left\langle K_{23}^{(N)} \right\rangle_{P'_b} = -\frac{3GM' \sin 2I' \cos \Omega'}{4a'^3 (1 - e'^2)^{3/2}}. \quad (35)$$

For the post-Newtonian gravitoelectric tidal field of M' , eq. (5) yields

$$\begin{aligned} \left\langle K_{11}^{(GE)} \right\rangle_{P'_b} = & -\frac{3G^2 M'^2 e'^2}{32c^2 a'^4 (1 - e'^2)^{5/2}} \left\{ 12 \cos 2I' + 2 \cos 2\omega' - 2 \cos 2I' \cos 2\omega' + \right. \\ & + \cos 2\Omega' \left[24 \sin^2 I' + 2 (\cos 2I' + 3) \cos 2\omega' \right] - \\ & \left. - 8 \cos I' \sin 2\Omega' \sin 2\omega' + 4 \right\}, \end{aligned} \quad (36)$$

$$\begin{aligned} \left\langle K_{22}^{(GE)} \right\rangle_{P'_b} = & \frac{3G^2 M'^2 e'^2}{32c^2 a'^4 (1 - e'^2)^{5/2}} \left\{ -12 \cos 2I' - 2 \cos 2\omega' + 2 \cos 2I' \cos 2\omega' + \right. \\ & + \cos 2\Omega' \left[24 \sin^2 I' + 2 (\cos 2I' + 3) \cos 2\omega' \right] - \\ & \left. - 8 \cos I' \sin 2\Omega' \sin 2\omega' - 4 \right\}, \end{aligned} \quad (37)$$

$$\left\langle K_{33}^{(\text{GE})} \right\rangle_{P'_b} = \frac{3G^2 M'^2 e'^2}{4c^2 a'^4 (1 - e'^2)^{5/2}} \left(\sin^2 I' \cos 2\omega' + 3 \cos 2I' + 1 \right), \quad (38)$$

$$\begin{aligned} \left\langle K_{12}^{(\text{GE})} \right\rangle_{P'_b} = & -\frac{3G^2 M'^2 e'^2}{16c^2 a'^4 (1 - e'^2)^{5/2}} \left\{ 4 \cos I' \cos 2\Omega' \sin 2\omega' + \right. \\ & \left. + \left[12 \sin^2 I' + (\cos 2I' + 3) \cos 2\omega' \right] \sin 2\Omega' \right\}, \end{aligned} \quad (39)$$

$$\left\langle K_{13}^{(\text{GE})} \right\rangle_{P'_b} = -\frac{3G^2 M'^2 e'^2}{4c^2 a'^4 (1 - e'^2)^{5/2}} \left\{ \sin I' \left[\cos \Omega' \sin 2\omega' + \cos I' (\cos 2\omega' - 6) \sin \Omega' \right] \right\}, \quad (40)$$

$$\left\langle K_{23}^{(\text{GE})} \right\rangle_{P'_b} = \frac{3G^2 M'^2 e'^2}{4c^2 a'^4 (1 - e'^2)^{5/2}} \left\{ \sin I' \left[\cos I' (\cos 2\omega' - 6) \cos \Omega' - \sin 2\omega' \sin \Omega' \right] \right\}. \quad (41)$$

In the limit of $e' \rightarrow 0$, eq. (36)-eq. (41) vanish.

For the post-Newtonian gravitomagnetic tidal field of M' due to \mathbf{J}' , from eq. (6) one obtains

$$\begin{aligned} \left\langle K_{11}^{(\text{GM})} \right\rangle_{P'_b} = & -\frac{3GJ'n'_b}{64c^2 a'^3 (1 - e'^2)^3} \left\{ 40e'^2 \sin 2\omega' \left(2\hat{k}'_z \cos 2I' + 3\hat{k}'_x \sin 2I' \sin \Omega' \right) \sin 2\Omega' + \right. \\ & + 5 \cos 2\omega' \left(12 \sin 3I' \left(\hat{k}'_x \sin \Omega' - \hat{k}'_y \cos \Omega' \right) \sin^2 \Omega' + \right. \\ & + \left. \sin I' \left(\hat{k}'_y \left(\cos \Omega' + 15 \cos 3\Omega' \right) - 3\hat{k}'_x \left(\sin \Omega' + 5 \sin 3\Omega' \right) \right) \right) e'^2 - \\ & - 20 \left(3e'^2 + 2 \right) \hat{k}'_z \cos 3I' - 4 \cos I' \left(5\hat{k}'_z \cos 2\omega' \left(6 \sin^2 I' + \right. \right. \\ & + \left. \left. \left(3 \cos 2I' + 1 \right) \cos 2\Omega' \right) e'^2 - 10\hat{k}'_y \sin I' \sin 2\omega' \left(\sin \Omega' - 3 \sin 3\Omega' \right) e'^2 + \right. \\ & \left. \left. + \left(3e'^2 + 2 \right) \hat{k}'_z \left(20 \cos 2\Omega' \sin^2 I' + 3 \right) \right) \right\} + \end{aligned}$$

$$\begin{aligned}
& + 2 \left(3e'^2 + 2 \right) \left(20 \sin 3I' \left(\hat{k}'_y \cos \Omega' - \hat{k}'_x \sin \Omega' \right) \sin^2 \Omega' + \right. \\
& \left. + \sin I' \left(\hat{k}'_y \left(\cos \Omega' + 15 \cos 3\Omega' \right) - 3\hat{k}'_x \left(\sin \Omega' + 5 \sin 3\Omega' \right) \right) \right) \Bigg\}, \quad (42)
\end{aligned}$$

$$\begin{aligned}
\left\langle K_{22}^{(\text{GM})} \right\rangle_{P'_b} = & - \frac{3GJ'n'_b}{64c^2a'^3 (1 - e'^2)^3} \left\{ 20e'^2 \hat{k}'_x \left(\cos \Omega' + 3 \cos 3\Omega' \right) \sin 2I' \sin 2\omega' - \right. \\
& - 80e'^2 \hat{k}'_z \cos 2I' \sin 2\omega' \sin 2\Omega' - 20 \left(3e'^2 + 2 \right) \hat{k}'_z \cos 3I' + \\
& + 20 \left(3e'^2 \cos 2\omega' - 6e'^2 - 4 \right) \cos^2 \Omega' \sin 3I' \left(\hat{k}'_x \sin \Omega' - \hat{k}'_y \cos \Omega' \right) + \\
& + 4e'^2 \cos I' \left(5\hat{k}'_z \cos 2\omega' \left(\left(3 \cos 2I' + 1 \right) \cos 2\Omega' - 6 \sin^2 I' \right) + \right. \\
& + 120e'^2 \hat{k}'_y \cos^2 \Omega' \sin I' \sin 2\omega' \sin \Omega' + \\
& + \left. \left(3e'^2 + 2 \right) \hat{k}'_z \left(20 \cos 2\Omega' \sin^2 I' - 3 \right) \right) + \\
& + \left(5e'^2 \cos 2\omega' + 6e'^2 + 4 \right) \sin I' \left(3\hat{k}'_y \cos \Omega' - \right. \\
& \left. - 15\hat{k}'_y \cos 3\Omega' - \hat{k}'_x \sin \Omega' + 15\hat{k}'_x \sin 3\Omega' \right) \Bigg\}, \quad (43)
\end{aligned}$$

$$\begin{aligned}
\left\langle K_{33}^{(\text{GM})} \right\rangle_{P'_b} = & - \frac{3GJ'n'_b}{16c^2a'^3 (1 - e'^2)^3} \left\{ -20e'^2 \sin 2I' \sin 2\Omega' \left(\hat{k}'_x \cos \Omega' + \hat{k}'_y \sin \Omega' \right) + \right. \\
& + 5e'^2 \cos 2\Omega' \left(12\hat{k}'_z \cos I' \sin^2 I' + \hat{k}'_y \cos \Omega' \left(3 \sin 3I' - \sin I' \right) + \right.
\end{aligned}$$

$$\begin{aligned}
& + \hat{k}'_x \left(\sin I' - 3 \sin 3I' \right) \sin \Omega' \Big) + 2 \left(3e'^2 + 2 \right) \left(3\hat{k}'_z \cos I' + 5\hat{k}'_z \cos 3I' - \right. \\
& \left. - \left(\sin I' + 5 \sin 3I' \right) \left(\hat{k}'_y \cos \Omega' - \hat{k}'_x \sin \Omega' \right) \right) \Big\} , \tag{44}
\end{aligned}$$

$$\begin{aligned}
\left\langle K_{12}^{(\text{GM})} \right\rangle_{P'_b} = & - \frac{3GJ'n'_b}{64c^2a'^3(1-e'^2)^3} \left\{ -80e'^2 \hat{k}'_z \cos 2I' \cos 2\Omega' \sin 2\omega' + \right. \\
& + 20e'^2 \sin 2I' \sin 2\omega' \left(\hat{k}'_y \cos \Omega' + 3\hat{k}'_y \cos 3\Omega' + \hat{k}'_x \left(\sin \Omega' - 3 \sin 3\Omega' \right) \right) - \\
& - 10\hat{k}'_z \cos 3I' \left(3e'^2 \cos 2\omega' - 6e'^2 - 4 \right) \sin 2\Omega' - \\
& - 10\hat{k}'_z \cos I' \left(5e'^2 \cos 2\omega' + 6e'^2 + 4 \right) \sin 2\Omega' + \\
& + 10 \left(3e'^2 \cos 2\omega' - 6e'^2 - 4 \right) \sin 3I' \left(\hat{k}'_y \cos \Omega' - \hat{k}'_x \sin \Omega' \right) \sin 2\Omega' + \\
& + \left(5e'^2 \cos 2\omega' + 6e'^2 + 4 \right) \sin I' \left(\hat{k}'_x \cos \Omega' + 15\hat{k}'_x \cos 3\Omega' - \right. \\
& \left. - \hat{k}'_y \sin \Omega' + 15\hat{k}'_y \sin 3\Omega' \right) \Big\} , \tag{45}
\end{aligned}$$

$$\begin{aligned}
\left\langle K_{13}^{(\text{GM})} \right\rangle_{P'_b} = & \frac{3GJ'n'_b}{32c^2a'^3(1-e'^2)^3} \left\{ -10e'^2 \hat{k}'_z \cos 2\omega' \left(\sin I' - 3 \sin 3I' \right) \sin \Omega' + \right. \\
& + 40e'^2 \sin 2\omega' \left(\hat{k}'_z \cos \Omega' \sin 2I' + \cos 2I' \left(\hat{k}'_y \cos 2\Omega' - \hat{k}'_x \sin 2\Omega' \right) \right) - \\
& - 4 \left(3e'^2 + 2 \right) \hat{k}'_z \left(\sin I' + 5 \sin 3I' \right) \sin \Omega' +
\end{aligned}$$

$$\begin{aligned}
& + 5e'^2 \cos 3I' \left(3 \cos 2\omega' \left(\hat{k}'_x \cos 2\Omega' + \hat{k}'_y \sin 2\Omega' \right) + 2 \left(3e'^2 + 2 \right) \hat{k}'_x \right) + \\
& + e'^2 \cos I' \left(5 \cos 2\omega' \left(12\hat{k}'_x \sin^2 I' + 5\hat{k}'_x \cos 2\Omega' + 5\hat{k}'_y \sin 2\Omega' \right) + \right. \\
& \left. + 2 \left(3e'^2 + 2 \right) \left(20 \left(\hat{k}'_x \cos 2\Omega' + \hat{k}'_y \sin 2\Omega' \right) \sin^2 I' + 3\hat{k}'_x \right) \right) \Bigg\}, \quad (46)
\end{aligned}$$

$$\begin{aligned}
\left\langle K_{23}^{(\text{GM})} \right\rangle_{P'_b} = & - \frac{3GJ'_b n'_b}{64c^2 a'^3 (1 - e'^2)^3} \left\{ 10e'^2 \cos 2\omega' \left(-12\hat{k}'_y \cos I' \sin^2 I' + \right. \right. \\
& + \hat{k}'_y \left(5 \cos I' + 3 \cos 3I' \right) \cos 2\Omega' - 2\hat{k}'_z \cos \Omega' \left(\sin I' - 3 \sin 3I' \right) - \\
& - \hat{k}'_x \left(5 \cos I' + 3 \cos 3I' \right) \sin 2\Omega' \Bigg) - 80e'^2 \sin 2\omega' \left(\hat{k}'_z \sin 2I' \sin \Omega' + \right. \\
& + \cos 2I' \left(\hat{k}'_x \cos 2\Omega' + \hat{k}'_y \sin 2\Omega' \right) \Bigg) + 4 \left(3e'^2 + 2 \right) \left(-5\hat{k}'_y \cos 3I' - \right. \\
& - 2\hat{k}'_z \cos \Omega' \left(\sin I' + 5 \sin 3I' \right) + \\
& \left. \left. + \cos I' \left(20 \sin^2 I' \left(\hat{k}'_y \cos 2\Omega' - \hat{k}'_x \sin 2\Omega' \right) - 3\hat{k}'_y \right) \right) \right\}. \quad (47)
\end{aligned}$$

Note that eq. (42)-eq. (47) have a general validity since they are restricted neither to any specific spatial orientation of the spin axis of M' nor to circular and/or equatorial orbits of M about M' .

It should be remarked that indirect, mixed effects of order $\mathcal{O}(c^{-2})$ arise, in principle, also from the Newtonian tidal matrix of eq. (4) when the post-Newtonian effects of the field of M' onto the orbital motion of M are taken into account [13]. The same holds also with J'_2 , accounting for possible deviations of M' from spherical symmetry at the Newtonian level itself. The calculation of such further effects, whose size may be comparable with that of the direct ones, is beyond the scope of this paper.

4 POSSIBLE SCENARIOS OF INTEREST FOR EMPIRICAL TESTS

Forthcoming space-based missions to astronomical bodies orbiting large primaries such as the Sun and Jupiter, in conjunction with expected progresses in interplanetary tracking techniques [34, 35, 36], may, in principle, represent an opportunity to put on the test the post-Newtonian tidal effects calculated in the previous Sections.

Let us consider the forthcoming BepiColombo³ [20] and JUICE⁴ [21] missions targeted to Mercury and the Jovian natural satellite Ganymede, respectively; in the following, the symbol ι will be adopted for the inclinations of the orbital planes of such spacecrafts to the equators of the orbited bodies. A probe named Mercury Planetary Orbiter⁵ (MPO) [37] is planned to be released in a polar ($\iota_{\text{MPO}} = 90$ deg), elliptical orbit ($e_{\text{MPO}} = 0.16$) around Mercury with an orbital period of approximately 2.3 h ($a_{\text{MPO}} = 3,394$ km). The nominal science duration is one year, with a possible extension of another year. Importantly, orbital maneuvers to change the attitude of the spacecraft are scheduled every about 44 d, so that long smooth orbital arcs should be available. The JUICE mission [21] to the Jovian system will culminate in a dedicated orbital tour around Ganymede which should encompass a 30 d science phase during which JUICE will orbit⁶ the satellite in a polar, circular low path with an altitude as little as $h = 200$ km. Table 1 summarizes the characteristic frequencies of both the scenarios considered, showing that our results of the previous Sections are applicable to them. In Table 2, we maximize the values of the post-Newtonian tidal perturbations for both MPO and JGO with respect to their unknown node Ω and pericenter ω . We do the same also for some of the most important competing Newtonian and post-Newtonian orbital perturbations. It can be noticed that, while for MPO the gravitoelectric tidal effects are larger than the gravitomagnetic ones, the situation is reversed for JGO because of its scheduled zero eccentricity. As far as the magnitudes of the tidal effects are concerned, they are larger for JGO; its gravitomagnetic tidal precessions reach the $\approx 10^{-1} - 10^{-2}$ mas yr⁻¹ level. In principle, a rate of ≈ 0.5 mas yr⁻¹, which naively corresponds to a range-rate as little as ≈ 0.05 mm s⁻¹ at the distance of Jupiter from us, might be detectable with the expected improvement down to 0.01 mm s⁻¹ at 60 s integration time in the Doppler range-rate techniques from the ASTRA study [36]. Unfortunately, such figures are too small if compared with those of the competing orbital effects. Suffice it to say that the oblateness of Ganymede is presently known with a relative uncertainty of the order of just $\sigma_{J_2}/J_2 = 2 \times 10^{-2}$ [38]. An improvement of 6 – 7 orders of magnitude is beyond the goals of the JUICE mission itself [21]. Strictly speaking, such considerations hold only for the direct post-Newtonian tidal effects calculated in the previous Sections; the total sensitivity budget should account for the indirect, mixed effects of order $\mathcal{O}(c^{-2})$ as well.

In principle, a rather unconventional possibility could be the realization of an artificial mini-planetary system to be carried onboard a drag-free spacecraft orbiting, say, the Earth; such an idea was already proposed in the past to accurately measure the Newtonian constant of gravitation G [48, 49, 50, 51, 52, 53, 54], and, more recently, to put on the test the MOND theory [55]. The conditions of validity of the present analysis could be fulfilled, e.g., by placing a non-rotating sphere made of tungsten with density $\rho_{\text{W}} = 19.6$ g cm⁻³ and $M = 28$ kg, $R = 7$ cm inside a drag-free spacecraft orbiting the Earth in some suitably chosen High Earth Orbit (HEO). By assuming, say, $a = 10$ cm for the test particle orbiting the tungsten sphere and a geostationary orbit with $a' = 42,164$ km for the spacecraft, it would be possible to obtain

$$\frac{n'_b}{n_b} = 5 \times 10^{-2}. \quad (48)$$

³ See also <http://sci.esa.int/bepicolombo/> on the Internet.

⁴ See also <http://sci.esa.int/juice/> on the Internet.

⁵ See <http://sci.esa.int/bepicolombo/48872-spacecraft/> on the Internet.

⁶ See <http://sci.esa.int/juice/50074-scenario-operations/> on the Internet.

Table 1. Some characteristic orbital frequencies, in s^{-1} , of the Sun-Mercury-Mercury Planetary Orbiter (MPO) and of the Jupiter-Ganymede-Jupiter Ganymede Orbiter (JGO) [21] systems. For JGO, we considered the planned 30 days phase of the Ganymede tour to be spent in a low altitude (200 km) circular orbit (<http://sci.esa.int/juice/50074-scenario-operations/>). For the sake of simplicity, the Newtonian and the post-Newtonian precessions due to the oblateness J_2 and to the angular momentum J of the primaries were calculated in equatorial coordinate systems. Indeed, while both MPO and JGO will move along polar trajectories at $\iota = 90$ deg to the equators of their primaries, the orbits of Mercury and Ganymede lie almost in the equatorial planes of the Sun ($\iota_{\text{☿}} = 3.38$ deg) and of Jupiter ($\iota_{\text{Gan}} = 0.20$ deg), respectively. For Ganymede, the value $J_2^{\text{Gan}} = 1.27 \times 10^{-4}$ [38] was adopted, while its angular momentum $J^{\text{Gan}} = 3 \times 10^{30} \text{ kg m}^2 \text{ s}^{-1}$ was inferred from the values of its mass, equatorial radius and normalized polar moment of inertia [38]. For Mercury, we assumed $J_2^{\text{☿}} = 1.92 \times 10^{-5}$ [39], while its angular momentum $J^{\text{☿}} = 8.4 \times 10^{29} \text{ kg m}^2 \text{ s}^{-1}$ was obtained from the latest determinations of its equatorial radius [40] and normalized polar moment of inertia [41]. For the angular momentum and the oblateness of the Sun and of Jupiter, we assumed $J^{\odot} = 1.90 \times 10^{41} \text{ kg m}^2 \text{ s}^{-1}$ [42], $J_2^{\odot} = 2.1 \times 10^{-7}$ [43], and $J^{\text{Jup}} = 6.9 \times 10^{38} \text{ kg m}^2 \text{ s}^{-1}$ [44], $J_2^{\text{Jup}} = 1.469 \times 10^{-2}$ [45], respectively.

Frequency (Sun-Mercury-MPO)	Value (s^{-1})	Frequency (Jupiter-Ganymede-JGO)	Value (s^{-1})
$\dot{\omega}_{\text{☿}}^{(\text{GM})}$	1×10^{-18}	$\dot{\Omega}_{\text{Gan}}^{(\text{GM})}$	8×10^{-16}
$\dot{\omega}_{\text{☿}}^{(J_2^{\odot})}$	8×10^{-17}	$\dot{\omega}_{\text{Gan}}^{(J_2^{\text{Jup}})}$	3×10^{-9}
$\dot{\omega}_{\text{☿}}^{(\text{GE})}$	7×10^{-14}	$\dot{\omega}_{\text{Gan}}^{(\text{GE})}$	4×10^{-14}
$n_{\text{☿}}$	8×10^{-7}	n_{Gan}	1×10^{-5}
$\dot{\Omega}_{\text{MPO}}^{(\text{GM})}$	3×10^{-17}	$\dot{\Omega}_{\text{JGO}}^{(\text{GM})}$	2×10^{-16}
$\dot{\omega}_{\text{MPO}}^{(\text{GE})}$	2×10^{-13}	$\dot{\omega}_{\text{JGO}}^{(\text{GE})}$	8×10^{-14}
$\left \dot{\omega}_{\text{MPO}}^{(J_2^{\text{☿}})} \right $	6×10^{-9}	$\left \dot{\omega}_{\text{JGO}}^{(J_2^{\text{Gan}})} \right $	5×10^{-8}
n_{MPO}	7×10^{-4}	n_{JGO}	6×10^{-4}

Moreover, the local dynamics of such a spaceborne artificial planetary system would be practically free from systematic non-Keplerian gravitational perturbations due to M . Indeed, careful manufacturing of the sphere would allow to make its oblateness negligible; the post-Newtonian gravitoelectric pericenter precession would be completely irrelevant being as little as $\dot{\omega} \approx 10^{-12} \text{ mas yr}^{-1}$. Table 3 summarizes the nominal maximum values of the precessions of a “planet” orbiting the aforementioned tungsten sphere along a circular path perpendicular to the Earth’s equator in a spacecraft following a highly eccentric polar orbit around the Earth. In fact, the size of the post-Newtonian tidal effects are not negligible. Nonetheless, the tidal precessions of Newtonian origin would overwhelm them since the gravitational parameter GM'

Table 2. Maximum nominal values of the direct orbital rates of change $\dot{\Psi}$ of MPO and JGO, averaged over both P_b and P'_b , induced by the post-Newtonian gravitoelectric and gravitomagnetic tidal field of M' , by its Newtonian tidal field, and by some competing Newtonian and post-Newtonian perturbations due to the deviation from spherical symmetry of the field of M . The values of Ω_{\max} and ω_{\max} , which are different for each orbital effect considered, are not reported. The units for the precessions are milliarcseconds per year (mas yr⁻¹), apart from the eccentricity e whose rate of change is expressed in s⁻¹. The Newtonian J_2 and the post-Newtonian J orbital precessions for a generic orientation of the spin axis of M were retrieved from [46]. The mean equinox and the mean equatorial plane of the Earth at the epoch J2000.0 of the International Celestial Reference Frame (ICRF) were adopted for both \mathcal{K} and \mathcal{K}' . The orientations of the spin axes with respect to the ICRF were retrieved from [47].

$\dot{\Psi}$ (MPO)	Value	$\dot{\Psi}$ (JGO)	Value
$\dot{e}^{(\text{tid GM})}$	$1 \times 10^{-21} \text{ s}^{-1}$	$\dot{e}^{(\text{tid GM})}$	0 s^{-1}
$\dot{I}^{(\text{tid GM})}$	$4 \times 10^{-6} \text{ mas yr}^{-1}$	$\dot{I}^{(\text{tid GM})}$	0.01 mas yr^{-1}
$\dot{\Omega}^{(\text{tid GM})}$	$1 \times 10^{-5} \text{ mas yr}^{-1}$	$\dot{\Omega}^{(\text{tid GM})}$	0.07 mas yr^{-1}
$\dot{\omega}^{(\text{tid GM})}$	$8 \times 10^{-5} \text{ mas yr}^{-1}$	$\dot{\omega}^{(\text{tid GM})}$	0.54 mas yr^{-1}
$\dot{e}^{(\text{tid GE})}$	$1 \times 10^{-18} \text{ s}^{-1}$	$\dot{e}^{(\text{tid GE})}$	0 s^{-1}
$\dot{I}^{(\text{tid GE})}$	$0.0025 \text{ mas yr}^{-1}$	$\dot{I}^{(\text{tid GE})}$	$8 \times 10^{-7} \text{ mas yr}^{-1}$
$\dot{\Omega}^{(\text{tid GE})}$	$0.0084 \text{ mas yr}^{-1}$	$\dot{\Omega}^{(\text{tid GE})}$	$2 \times 10^{-6} \text{ mas yr}^{-1}$
$\dot{\omega}^{(\text{tid GE})}$	$0.0458 \text{ mas yr}^{-1}$	$\dot{\omega}^{(\text{tid GE})}$	$1 \times 10^{-5} \text{ mas yr}^{-1}$
$\dot{I}^{(\text{GM})}$	$0.1038 \text{ mas yr}^{-1}$	$\dot{I}^{(\text{GM})}$	0.59 mas yr^{-1}
$\dot{\Omega}^{(\text{GM})}$	$0.1907 \text{ mas yr}^{-1}$	$\dot{\Omega}^{(\text{GM})}$	1.24 mas yr^{-1}
$\dot{\omega}^{(\text{GM})}$	$0.2075 \text{ mas yr}^{-1}$	$\dot{\omega}^{(\text{GM})}$	1.18 mas yr^{-1}
$\dot{\omega}^{(\text{GE})}$	$1087.78 \text{ mas yr}^{-1}$	$\dot{\omega}^{(\text{GE})}$	$499.6 \text{ mas yr}^{-1}$
$\dot{e}^{(\text{tid N})}$	$2.9 \times 10^{-10} \text{ s}^{-1}$	$\dot{e}^{(\text{tid N})}$	0 s^{-1}
$\dot{I}^{(\text{tid N})}$	$6.7 \times 10^5 \text{ mas yr}^{-1}$	$\dot{I}^{(\text{tid N})}$	$7 \times 10^7 \text{ mas yr}^{-1}$
$\dot{\Omega}^{(\text{tid N})}$	$2 \times 10^6 \text{ mas yr}^{-1}$	$\dot{\Omega}^{(\text{tid N})}$	$3 \times 10^8 \text{ mas yr}^{-1}$
$\dot{\omega}^{(\text{tid N})}$	$1.3 \times 10^7 \text{ mas yr}^{-1}$	$\dot{\omega}^{(\text{tid N})}$	$2 \times 10^9 \text{ mas yr}^{-1}$
$\dot{I}^{(J_2)}$	$8 \times 10^6 \text{ mas yr}^{-1}$	$\dot{I}^{(J_2)}$	$6 \times 10^7 \text{ mas yr}^{-1}$
$\dot{\Omega}^{(J_2)}$	$3 \times 10^7 \text{ mas yr}^{-1}$	$\dot{\Omega}^{(J_2)}$	$2.7 \times 10^8 \text{ mas yr}^{-1}$
$\dot{\omega}^{(J_2)}$	$4 \times 10^7 \text{ mas yr}^{-1}$	$\dot{\omega}^{(J_2)}$	$3.5 \times 10^8 \text{ mas yr}^{-1}$

of the Earth is currently known with a 2×10^{-9} relative accuracy [56], insufficient by 1 – 2 orders of magnitude for our purposes.

5 OVERVIEW AND CONCLUSIONS

We looked at the direct long-term orbital rates of change occurring within a local gravitationally bound two-body system as gradiometers to potentially detect post-Newtonian tidal effects due to its slow motion in the external field of a distant third body. We also assumed that the characteristic orbital frequencies of the internal dynamics of the local binary are quite smaller than the frequency of its orbital motion around the external source. We obtained general analytical expressions valid for arbitrary orbital configurations and for a generic orientation of the spin axis of the external body. Future work should be devoted to the calculation of the indirect, mixed post-Newtonian effects arising from the interplay between the Newtonian tidal matrix and the post-Newtonian orbital motion of the binary in the external field.

Table 3. Maximum nominal values of the direct orbital rates of change $\dot{\Psi}$ (here, $\varpi \doteq \Omega + \omega$ is the longitude of the pericenter) of a member of a spaceborne artificial mini-planetary system orbiting a tungsten sphere of mass $M = 28$ kg and radius $R = 7$ cm in a circular orbit with $a = 10$ cm, $e = 0$, $I = 90$ deg averaged over both P_b and P'_b , induced by the post-Newtonian gravitoelectric and gravitomagnetic tidal field of the Earth and by its Newtonian tidal field. For the spacecraft hosting it we assumed an highly elliptical geostationary polar orbit characterized by $a' = 42,164$ km, $e' = 0.7$, $I' = \omega' = 90$ deg, $\Omega' = 0$ deg. The values of Ω_{\max} and ω_{\max} , which are different for each orbital effect considered, are not reported. The units for the precessions are milliarcseconds per year (mas yr⁻¹), apart from the eccentricity e whose rate of change is expressed in s⁻¹. The mean equinox and the mean equatorial plane of the Earth at the epoch J2000.0 of the International Celestial Reference Frame (ICRF) were adopted for both \mathcal{K} and \mathcal{K}' . The orientation of the Earth's spin axis with respect to the ICRF was retrieved from [47].

$\dot{\Psi}$	Value
$\dot{e}^{(\text{tid GM})}$	0 s ⁻¹
$\dot{I}^{(\text{tid GM})}$	0 mas yr ⁻¹
$\dot{\varpi}^{(\text{tid GM})}$	3.2 mas yr ⁻¹
$\dot{e}^{(\text{tid GE})}$	0 s ⁻¹
$\dot{I}^{(\text{tid GE})}$	32.8 mas yr ⁻¹
$\dot{\varpi}^{(\text{tid GE})}$	189.6 mas yr ⁻¹
$\dot{e}^{(\text{tid N})}$	0 s ⁻¹
$\dot{I}^{(\text{tid N})}$	2.6×10^{10} mas yr ⁻¹
$\dot{\varpi}^{(\text{tid N})}$	1.5×10^{11} mas yr ⁻¹

We applied our results to the future BepiColombo and JUICE man-made missions to Mercury and Ganymede, respectively. It turned out that that, although the expected improvements in interplanetary tracking may, perhaps, allow for a detection of the tidal effects we are interested in, especially for JUICE, the impact of several competing orbital effects of Newtonian and post-Newtonian origin, acting as sources of potential systematic errors, should be carefully considered.

Another possibility which, in principle, may be further pursued is the realization of an artificial mini-planetary system to be carried onboard an Earth-orbiting drag-free spacecraft. If, on the one hand, the post-Newtonian tidal precessions occurring in such a system may be relatively large, amounting to about $1 - 10^2$ mas yr⁻¹, on the other hand, the product of the Earth's mass times the Newtonian gravitational constant is currently known with insufficient accuracy to allow for an effective subtraction of the competing Newtonian tidal precessions.

REFERENCES

- [1] Will CM. The Confrontation Between General Relativity and Experiment. *Space Science Reviews* **148** (2009) 3–13. doi:10.1007/s11214-009-9541-6.
- [2] Mashhoon B. Tidal radiation. *The Astrophysical Journal* **216** (1977) 591–609. doi:10.1086/155500.
- [3] Chicone C, Mashhoon B. The generalized Jacobi equation. *Classical and Quantum Gravity* **19** (2002) 4231–4248. doi:10.1088/0264-9381/19/16/301.
- [4] Chicone C, Mashhoon B. Tidal dynamics in Kerr spacetime. *Classical and Quantum Gravity* **23** (2006) 4021–4033. doi:10.1088/0264-9381/23/12/002.
- [5] Kopeikin S, Efroimsky M, Kaplan G. *Relativistic Celestial Mechanics of the Solar System* (Wiley, New York) (2011).

- [6] Braginsky VB, Polnarev AG. Relativistic spin-quadrupole gravitational effect. *Journal of Experimental and Theoretical Physics Letters* **31** (1980) 415–418.
- [7] Mashhoon B, Theiss DS. Relativistic Tidal Forces and the Possibility of Measuring Them. *Physical Review Letters* **49** (1982) 1542–1545. doi:10.1103/PhysRevLett.49.1542.
- [8] Mashhoon B, Theiss DS. Gravitational influence of the rotation of the sun on the earth-moon system. *Physics Letters A* **115** (1986) 333–337. doi:10.1016/0375-9601(86)90625-0.
- [9] Gill E, Schastok J, Soffel MH, Ruder H. On the Mashhoon-Theiss “anomaly”. *Physical Review D* **39** (1989) 2441–2443. doi:10.1103/PhysRevD.39.2441.
- [10] Mashhoon B, Theiss DS. Relativistic lunar theory. *Nuovo Cimento B* **106** (1991) 545–571. doi:10.1007/BF02726789.
- [11] Mashhoon B, Theiss DS. Relativistic Effects in the Motion of the Moon. Lämmerzahl C, Everitt CWF, Hehl FW, editors, *Gyros, Clocks, Interferometers ...: Testing Relativistic Gravity in Space* (2001), *Lecture Notes in Physics, Berlin Springer Verlag*, vol. 562, 310.
- [12] Theiss DS. A general relativistic effect of a rotating spherical mass and the possibility of measuring it in a space experiment. *Physics Letters A* **109** (1985) 19–22. doi:10.1016/0375-9601(85)90382-2.
- [13] Mashhoon B, Paik HJ, Will CM. Detection of the gravitomagnetic field using an orbiting superconducting gravity gradiometer. Theoretical principles. *Physical Review D* **39** (1989) 2825–2838. doi:10.1103/PhysRevD.39.2825.
- [14] Paik HJ. Tests of general relativity in earth orbit using a superconducting gravity gradiometer. *Advances in Space Research* **9** (1989) 41–50. doi:10.1016/0273-1177(89)90006-9.
- [15] Theiss DS. A Gradiometer Experiment to Detect the Gravitomagnetic Field of the Earth. Ehlers J, Schäfer G, editors, *Relativistic Gravity Research* (1992), *Lecture Notes in Physics, Berlin Springer Verlag*, vol. 410, 131. doi:10.1007/3-540-56180-3.6.
- [16] Paik HJ. Detection of the gravitomagnetic field using an orbiting superconducting gravity gradiometer: principle and experimental considerations. *General Relativity and Gravitation* **40** (2008) 907–919. doi:10.1007/s10714-007-0582-4.
- [17] Li XQ, Shao MX, Paik HJ, Huang YC, Song TX, Bian X. Effects of satellite positioning errors and earths multipole moments in the detection of the gravitomagnetic field with an orbiting gravity gradiometer. *General Relativity and Gravitation* **46** (2014) 1–14. doi:10.1007/s10714-014-1737-8.
- [18] Brumberg VA, Kopeikin SM. Relativistic reference systems and motion of test bodies in the vicinity of the Earth. *Nuovo Cimento B* **103** (1989) 63–98. doi:10.1007/BF02888894.
- [19] Solomon SC, McNutt RL, Gold RE, Domingue DL. MESSENGER Mission Overview. *Space Science Reviews* **131** (2007) 3–39. doi:10.1007/s11214-007-9247-6.
- [20] Benkhoff J, van Casteren J, Hayakawa H, Fujimoto M, Laakso H, Novara M, et al. BepiColombo—Comprehensive exploration of Mercury: Mission overview and science goals. *Planetary and Space Science* **58** (2010) 2–20. doi:10.1016/j.pss.2009.09.020.
- [21] Grasset O, Dougherty MK, Coustenis A, Bunce EJ, Erd C, Titov D, et al. JUpiter ICy moons Explorer (JUICE): An ESA mission to orbit Ganymede and to characterise the Jupiter system. *Planetary and Space Science* **78** (2013) 1–21. doi:10.1016/j.pss.2012.12.002.
- [22] Einstein A. Erklärung der Perihelbewegung des Merkur aus der allgemeinen Relativitätstheorie. *Sitzungsberichte der Preußischen Akademie der Wissenschaften* **47** (1915) 831–839.
- [23] Nobili AM, Will CM. The real value of Mercury’s perihelion advance. *Nature* **320** (1986) 39–41. doi:10.1038/320039a0.
- [24] Lense J, Thirring H. Über den Einfluß der Eigenrotation der Zentralkörper auf die Bewegung der Planeten und Monde nach der Einsteinschen Gravitationstheorie. *Physikalische Zeitschrift* **19** (1918) 156–163.
- [25] Bertotti B, Farinella P, Vokrouhlický D. *Physics of the Solar System* (Kluwer Academic Press, Dordrecht) (2003).
- [26] Fermi E. Sopra i fenomeni che avvengono in vicinanza di una linea oraria. *Rendiconti dei Lincei* **31** (1922) 21–23; 51–52.
- [27] Levi-Civita T. Sur l’ écart géodésique. *Mathematische Annalen* **97** (1926) 291–320.
- [28] Synge JL. On the Geometry of Dynamics. *Royal Society of London Philosophical Transactions Series A* **226** (1927) 31–106. doi:10.1098/rsta.1927.0002.

- [29] de Sitter W. On Einstein's theory of gravitation and its astronomical consequences. Second paper. *Monthly Notices of the Royal Astronomical Society* **77** (1916) 155–184.
- [30] Fokker AD. The geodesic precession: a consequence of Einstein's theory of gravitation. *Koninklijke Nederlandsche Akademie van Wetenschappen, Proceedings* **23** (1921) 729738.
- [31] Burns JA. Elementary derivation of the perturbation equations of celestial mechanics. *American Journal of Physics* **44** (1976) 944–949. doi:10.1119/1.10237.
- [32] Brumberg VA. *Essential Relativistic Celestial Mechanics* (Adam Hilger, Bristol) (1991).
- [33] Will CM. Incorporating post-Newtonian effects in N-body dynamics. *Physical Review D* **89** (2014) 044043. doi:10.1103/PhysRevD.89.044043.
- [34] Iess L, Asmar S. Probing Space-Time in the Solar System: from Cassini to Bepicolombo. *International Journal of Modern Physics D* **16** (2007) 2117–2126. doi:10.1142/S0218271807011449.
- [35] Iess L, Asmar S, Tortora P. MORE: An advanced tracking experiment for the exploration of Mercury with the mission BepiColombo. *Acta Astronautica* **65** (2009) 666–675.
- [36] Iess L, Di Benedetto M, James N, Mercolino M, Simone L, Tortora P. Astra: Interdisciplinary study on enhancement of the end-to-end accuracy for spacecraft tracking techniques. *Acta Astronautica* **94** (2014) 699–707. doi:10.1016/j.actaastro.2013.06.011.
- [37] Ashby N, Bender PL, Wahr JM. Future gravitational physics tests from ranging to the BepiColombo Mercury planetary orbiter. *Physical Review D* **75** (2007) 022001. doi:10.1103/PhysRevD.75.022001.
- [38] Anderson JD, Lau EL, Sjogren WL, Schubert G, Moore WB. Gravitational constraints on the internal structure of Ganymede. *Nature* **384** (1996) 541–543. doi:10.1038/384541a0.
- [39] Smith DE, Zuber MT, Phillips RJ, Solomon SC, Neumann GA, Lemoine FG, et al. The equatorial shape and gravity field of Mercury from MESSENGER flybys 1 and 2. *Icarus* **209** (2010) 88–100. doi:10.1016/j.icarus.2010.04.007.
- [40] Byrne PK, Klimczak C, Celâl Şengör AM, Solomon SC, Watters TR, Hauck SA II. Mercury's global contraction much greater than earlier estimates. *Nature Geoscience* **7** (2014) 301–307. doi:10.1038/ngeo2097.
- [41] Margot JL, Peale SJ, Solomon SC, Hauck SA II, Ghigo FD, Jurgens RF, et al. Mercury's moment of inertia from spin and gravity data. *Journal of Geophysical Research (Planets)* **117** (2012) E00L09. doi:10.1029/2012JE004161.
- [42] Pijpers FP. Helioseismic determination of the solar gravitational quadrupole moment. *Monthly Notices of the Royal Astronomical Society* **297** (1998) L76–L80. doi:10.1046/j.1365-8711.1998.01801.x.
- [43] Folkner WM, Williams JG, Boggs DH, Park RS, Kuchynka P. The Planetary and Lunar Ephemerides DE430 and DE431. *Interplanetary Network Progress Report* **196** (2014) C1.
- [44] Soffel M, Klioner SA, Petit G, Wolf P, Kopeikin SM, Bretagnon P, et al. The IAU 2000 Resolutions for Astrometry, Celestial Mechanics, and Metrology in the Relativistic Framework: Explanatory Supplement. *The Astronomical Journal* **126** (2003) 2687–2706. doi:10.1086/378162.
- [45] Jacobson RA. JUP230 orbit solution. *NASA Technical Memorandum* (2003).
- [46] Iorio L. Perturbed stellar motions around the rotating black hole in Sgr A* for a generic orientation of its spin axis. *Physical Review D* **84** (2011) 124001. doi:10.1103/PhysRevD.84.124001.
- [47] Seidelmann PK, Archinal BA, A'Hearn MF, Conrad A, Consolmagno GJ, Hestroffer D, et al. Report of the IAU/IAG Working Group on cartographic coordinates and rotational elements: 2006. *Celestial Mechanics and Dynamical Astronomy* **98** (2007) 155–180. doi:10.1007/s10569-007-9072-y.
- [48] Nobili AM, Milani A, Farinella P. Testing Newtonian gravity in space. *Physics Letters A* **120** (1987) 437–441. doi:10.1016/0375-9601(87)90105-8.
- [49] Nobili AM, Milani A, Farinella P. The orbit of a space laboratory for the measurement of G. *The Astronomical Journal* **95** (1988) 576–578. doi:10.1086/114657.
- [50] Nobili AM, Milani A, Polacco E, Roxburgh IW, Barlier F, Aksnes K, et al. The Newton mission - a proposed manmade planetary system in space to measure the gravitational constant. *ESA Journal* **14** (1990) 389–408.
- [51] Keyser PT. Perturbative forces on an artificial binary for measuring G. *Physics Letters A* **167** (1992) 29–31. doi:10.1016/0375-9601(92)90621-R.

-
- [52] Sanders AJ, Deeds WE. Proposed new determination of the gravitational constant G and tests of Newtonian gravitation. *Physical Review D* **46** (1992) 489–504. doi:10.1103/PhysRevD.46.489.
- [53] Sanders AJ, Alexeev AD, Allison SW, Bronnikov KA, Campbell JW, Cates MR, et al. Project SEE (Satellite Energy Exchange): proposal for space-based gravitational measurements. *Measurement Science and Technology* **10** (1999) 514–524. doi:10.1088/0957-0233/10/6/317.
- [54] Sanders AJ, Alexeev AD, Allison SW, Antonov V, Bronnikov KA, Campbell JW, et al. Project SEE (Satellite Energy Exchange): an international effort to develop a space-based mission for precise measurements of gravitation. *Classical and Quantum Gravity* **17** (2000) 2331–2346. doi:10.1088/0264-9381/17/12/305.
- [55] Sahni V, Shtanov Y. Apsis: An Artificial Planetary System in Space to Probe Extra-Dimensional Gravity and MOND. *International Journal of Modern Physics D* **17** (2008) 453–466. doi:10.1142/S0218271808012127.
- [56] Petit G, Luzum B. IERS Conventions (2010). *IERS Technical Note* **36** (2010) 1.